



A combinatorial approach to certain topological spaces based on minimum complement S-approximation spaces

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Abstract

An S-approximation space is a novel approach to study systems with uncertainty that are not expressible in terms of inclusion relations. In this work, we further examined these spaces, mostly from a topological point of view by a combinatorial approach. This work also identifies a subclass of these approximation spaces, called S_{MC} -approximations. Topological properties of this subclass are investigated and finally, the topologies formed by S_{MC} -approximations are enumerated up to homeomorphism.

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1 Introduction

In almost all real-life applications we should handle uncertainty. In non-crisp sets, uncertainty is characterized by boundary regions, non-empty subsets of the universe where nothing can be said about their element memberships. Approximation is one of the ways to deal with these uncertainties. In classical set theory, a subset A of a universe U induces a partition $\{A, U - A\}$ on that universe. This partition might be interpreted as a knowledge about elements of U , i.e. elements of A are indiscernible. The same thing holds for elements of $U - A$. This can be generalized to any partition \mathcal{P} of U , supposing that elements in the same equivalence class of \mathcal{P} are not distinguishable but those in different classes are. In consequence, for a subset A of U , the problem of whether x belongs to A or not, with respect to knowledge \mathcal{P} , may become undecidable, i.e. we may have indiscernible elements, with respect to \mathcal{P} , which are or are not members of A . To cope with such uncertainty, a number of tools are invented such as Dempster-Shafer theory of evidence [29], theory of fuzzy sets [46, 45, 44, 43], and theory of rough sets

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[20, 19, 18]. Rough set theory and Dempster-Shafer theory of evidence are two independent approaches for handling uncertainty, but there is an important resemblance between the two. More formally, lower and upper approximations of rough set theory correspond to the inner and outer reductions from Dempster-Shafer theory[5].

Since their introduction in 1980s [18], rough sets have been applied to many different areas, such as discovering data patterns, a core subject of data mining, and dealing with incomplete information systems[21, 23]. Studying rough set and its generalizations together with topology has been an interesting research topic, as discussed in [28, 10, 25] and the connection between rough set theory and topology was found early in the framework of topology of partitions. Basic dependence of rough sets on certain topological spaces made this discovery not too unexpected[16, 26]. L. Polkowski implemented topological spaces using rough sets which were based on information systems [28]. A. Skowron in 1988 and A. Wiweger at the same time, but independently, discussed it on Z. Pawlak's rough sets. The relationship between the modified sets, topological spaces and rough sets based on pre-order was considered by J. Kortelainen in 1994[8]. This discussion was continued in [14, 15]. Analyzing the relation between generalized rough sets and topologies from different viewpoints is another interesting research area.

Rough set theory and its generalizations are all based on the inclusion relation [24, 22, 18, 19, 41, 40, 39, 42], which can be considered as a limitation. In this work, we use a new concept named *S*-approximation set which is proposed in [6]. This concept is independent from the inclusion relation and contains rough sets and their generalizations as special cases. It is also applied in three-way decision theory in [31], neighborhood systems in [32], fuzzy and intuitionistic fuzzy set theories [33].

Then we will study topological spaces built upon these approximations. Moreover, we will discuss homeomorphisms between such topologies and state the necessary and sufficient condition under which two such topologies are homeomorphic. It is well-known that homeomorphism is an equivalence relation over the class of all topologies. We will count the number of equivalence classes under homeomorphism over these topologies as well.

2 Preliminaries

In this section, we introduce the fundamental concepts of rough sets, generalizations of rough sets and topological spaces.

2.1 Basic Rough Set and its Generalizations

Let U be a non-empty finite set and $R \subseteq U \times U$, an equivalence relation on U . This relation partitions the set U into equivalence classes like $[x]_R$ which consists of all $y \in U$ such that xRy . Let X be a subset of U , then the set X can be approximated by equivalence classes of R by constructing the lower and upper approximations of X with respect to R , as is proposed by Z. Pawlak in [18] as follows

$$\underline{app}_R(X) = \{x \in U \mid [x]_R \subseteq X\},$$

$$\overline{app}_R(X) = \{x \in U \mid [x]_R \cap X \neq \emptyset\}.$$

If $\underline{app}_R(X) = \overline{app}_R(X)$, then the set X is called *definable* with respect to R , otherwise it is called a *rough set* with respect to R . The ordered pair $(\underline{app}_R(X), \overline{app}_R(X))$ is called the *approximation space* of X with respect to R .

Proposition 2.1 ([18]). *Let U be a non-empty finite set and $R \subseteq U \times U$ denotes an equivalence relation on U , then for every $X, Y \subseteq U$ the following properties hold:*

1. $\underline{app}_R(X) \subseteq X \subseteq \overline{app}_R(X)$,
2. $\underline{app}_R(U) = \overline{app}_R(U) = U$ and $\underline{app}_R(\emptyset) = \overline{app}_R(\emptyset) = \emptyset$,
3. $\overline{app}_R(X \cup Y) = \overline{app}_R(X) \cup \overline{app}_R(Y)$,
4. $\underline{app}_R(X \cap Y) = \underline{app}_R(X) \cap \underline{app}_R(Y)$,
5. $X \subseteq Y$ implies that $\underline{app}_R(X) \subseteq \underline{app}_R(Y)$,
6. $X \subseteq Y$ implies that $\overline{app}_R(X) \subseteq \overline{app}_R(Y)$,
7. $\underline{app}_R(X) \cup \underline{app}_R(Y) \subseteq \underline{app}_R(X \cup Y)$,
8. $\overline{app}_R(X \cap Y) \subseteq \overline{app}_R(X) \cap \overline{app}_R(Y)$,
9. $\underline{app}_R(X) = (\overline{app}_R(X^c))^c$ and equivalently $\overline{app}_R(X) = (\underline{app}_R(X^c))^c$.

Y. Yao's extension of Z. Pawlak's rough set is obtained by using an arbitrary relation, possibly not equivalence [41]. Let R be a binary relation on U . Then the ordered pair (U, R) is called a *generalized approximation space* based on the relation R . For $X \subseteq U$, the lower and upper approximations of set X are generalized as

$$\underline{app}_R(X) = \{x \in U \mid R(x) \subseteq X\},$$

and

$$\overline{app}_R(X) = \{x \in U \mid R(x) \cap X \neq \emptyset\},$$

where $R(x) = \{y \in U \mid (x, y) \in R\}$.

Proposition 2.2. [41] *Let U be a non-empty finite set and $R \subseteq U \times U$ an arbitrary relation on U , then for every $X, Y \subseteq U$, properties of proposition (2.1) are satisfied.*

There are also many other generalizations of rough set theory such as rough set models for incomplete information systems [34, 35, 9, 13], rough set models based on coverings [47, 1, 7] and rough fuzzy sets or fuzzy rough sets [4]. Moreover, these models can be generalized to the case of two universes [36, 37] based on the Shafer's compatibility view [30, 27], generalized rough fuzzy sets [38], and arbitrary relations [2].

2.2 Topological Spaces

In this section, we would briefly review basic concepts of topologies. A pair (X, τ) where X is a non-empty set and τ is a family of subsets of X containing \emptyset and X is called a *topology* if τ is closed under arbitrary union and finite intersection. Members of τ are called open sets and their complements are called closed sets. Let (X, τ_X) and (Y, τ_Y) be two topologies. A function $f : X \rightarrow Y$ is said to be *continuous* if for any open subset A in Y , $f^{-1}(A)$ is also open in X . Moreover, a bijective continuous map $\Phi : X \rightarrow Y$ is called a *homeomorphism* if Φ^{-1} is also continuous. If such a Φ exists, then (X, τ_X) and (Y, τ_Y) are called *homeomorphic* topologies. Homeomorphic topologies form equivalence classes over any set of topologies.

For every binary relation R over U , we can examine a topology generated by R . The *right neighborhood* is defined as $xR = \{y \in U | (x, y) \in R\}$, and the topology over U is denoted by (U, τ_R) , where $\tau_R = \{xR | x \in U\}$. For more details, refer to [12, 11].

2.3 S-approximation

S -approximation is a new mathematical approach to study approximation spaces [6]. This approach is proposed on the basis of the ideas of Dempster's multi-valued mappings [3], and has the Pawlak's rough set and its generalizations as special cases. These spaces are first proposed in [6] and are reviewed in this section.

Definition 2.3 ([6]). An S -approximation is the quadruple $G = (U, W, T, S)$ where U and W are finite non-empty sets, T is a mapping of the form $T : U \rightarrow P^*(W)$ and S is a mapping of the form $S : P^*(W) \times P^*(W) \rightarrow \{0, 1\}$.

For a non-empty subset X of W , the upper and lower approximations of X are defined as follows:

$$\overline{G}(X) = \{x \in U | S(T(x), X^c) = 0\},$$

and

$$\underline{G}(X) = \{x \in U | S(T(x), X) = 1\},$$

where X^c is the complement of X with respect to W .

2.3.1 $S_{\mathcal{M}}$ -approximations

There exists a know sub-class of S -approximation spaces which satisfy properties (3) to (10) of proposition (2.1), but are not inclusion-based. These properties are satisfied because their S relation satisfies the S -min condition, introduced in [6].

Definition 2.4 (S -min Condition [6]). Let $G = (U, W, T, S)$ be an S -approximation. We say that the relation $S : P^*(W) \times P^*(W) \rightarrow \{0, 1\}$ is a relation in $S_{\mathcal{M}}$ class if it satisfies

$$S(A, B \cap C) = \min \{S(A, B), S(A, C)\},$$

for arbitrary non-empty subsets A , B , and C of W . We also say an S -approximation $G = (U, W, T, S')$ is an $S_{\mathcal{M}}$ -approximation if S' belongs to the $S_{\mathcal{M}}$ class.

Remark 2.5. The inclusion relation does indeed belong to the $S_{\mathcal{M}}$ class but there are other non-inclusion relations in this class as well, cf. [6].

The following proposition is the counterpart of proposition (2.1) in $S_{\mathcal{M}}$ -approximation spaces.

Proposition 2.6. [6] *Let $G = (U, W, T, S)$ be an $S_{\mathcal{M}}$ -approximation. For all $A, B \subseteq W$ and $x \in U$, the following hold:*

1. $A \subseteq B$ implies that for all $X \subseteq W$, $S(X, B^c) \leq S(X, A^c)$,
2. $\max\{S(T(x), A), S(T(x), B)\} \leq S(T(x), A \cup B)$,
3. $\overline{G}(A \cup B) = \overline{G}(A) \cup \overline{G}(B)$,
4. $\underline{G}(A \cap B) = \underline{G}(A) \cap \underline{G}(B)$,
5. $A \subseteq B$ implies $\underline{G}(A) \subseteq \underline{G}(B)$,
6. $A \subseteq B$ implies $\overline{G}(A) \subseteq \overline{G}(B)$,
7. $\underline{G}(A) \cup \underline{G}(B) \subseteq \underline{G}(A \cup B)$,
8. $\overline{G}(A \cap B) \subseteq \overline{G}(A) \cap \overline{G}(B)$,
9. $\underline{G}(A) = (\overline{G}(A^c))^c$ and equivalently $\overline{G}(A) = (\underline{G}(A^c))^c$.

Interestingly, it is not always the case that $\underline{G}(A) \subseteq \overline{G}(A)$ in $S_{\mathcal{M}}$ -approximation spaces, although this property always holds in Pawlak's rough sets [6].

Example 2.7. [6] Suppose $G = (U, W, T, S)$ is an $S_{\mathcal{M}}$ -approximation where

$$S(A, B) = \begin{cases} 1 & A \cup B = W \\ 0 & \text{otherwise} \end{cases},$$

$U = \{a\}$, and $T(a) = W = \{1, 2\}$.

In this case $\overline{G}(\{1\}) = \{x \in U | T(x) \cup \{2\} \neq W\} = \emptyset$, while

$$\underline{G}(\{1\}) = \{x \in U | T(x) \cup \{1\} = W\} = \{a\},$$

so $\underline{G}(\{1\}) \not\subseteq \overline{G}(\{1\})$.

The structure of $S_{\mathcal{M}}$ -approximations plays an important role in understanding the topological structures which will be introduced in later sections, so we remind some results from [6].

Definition 2.8 ([6]). Let W be a non-empty finite set. A function $f : P^*(W) \rightarrow \{0, 1\}$ is said to be minimizing if for each $A, B \subseteq W$,

$$f(A \cap B) = \min \{f(A), f(B)\}.$$

Lemma 2.9. [6] Let $f : P^*(W) \rightarrow \{0, 1\}$ be a minimizing function. For each $A, B \subseteq W$, if $A \subseteq B$, then $f(A) \leq f(B)$.

Lemma 2.10. [6] Let $G = (U, W, T, S)$ be an $S_{\mathcal{M}}$ -approximation and $|W| = n$. We label the non-empty subsets of W as $\{A_1, \dots, A_{2^n-1}\}$. Then there exist minimizing functions $\{f_1, \dots, f_{2^n-1}\}$ of the form $f_i : P^*(W) \rightarrow \{0, 1\}$ such that for every $B \subseteq W$, we have $S(A_i, B) = f_i(B)$ for $1 \leq i \leq n$.

Lemma (2.10) leads us towards counting the number and finding the structure of minimizing f s.

Definition 2.11 ([6]). Let $f : P^*(W) \rightarrow \{0, 1\}$ be a minimizing function. A non-empty subset ω of the set W is called an atom of f if and only if $f(\omega) = 1$ and for each proper non-empty subset of ω such as η , $f(\eta) = 0$.

Proposition 2.12. [6] Let $f : P^*(W) \rightarrow \{0, 1\}$ be a minimizing function and ω_1 and ω_2 two non-identical atoms of f . Then $\omega_1 \cap \omega_2 = \emptyset$.

Proposition 2.13. [6] Let $f : P^*(W) \rightarrow \{0, 1\}$ be a minimizing function and Υ the set of all atoms of f . Then for a subset X of W , $f(X) = 1$ if and only if there exists $\omega \in \Upsilon$ such that $\omega \subseteq X$.

Proposition 2.14. [6] Let $f : P^*(W) \rightarrow \{0, 1\}$ be a minimizing function, Υ the set of all atoms of f and $|\Upsilon| \geq 2$. Then for each $x \in W$, $\{x\}$ is an atom of f .

By previous propositions, it is clear that we either have no atoms, or exactly one atom or an atom per element.

3 Topologies of $S_{\mathcal{MC}}$ -approximations

In this paper, we are interested in topological structures over a special class of $S_{\mathcal{M}}$ -approximations, where the S relation is extended to $P(W) \times P(W) \rightarrow \{0, 1\}$ and satisfies the complement condition, which is defined as:

$$S(A, B^c) = 1 - S(A, B), \quad (3.1)$$

for any $A, B \subseteq W$ and $S(A, \emptyset) = 0$. We use the notation $S_{\mathcal{MC}}$ to denote this class of $S_{\mathcal{M}}$ -approximations.

Lemma 3.1. Let $G = (U, W, T, S)$ be an $S_{\mathcal{MC}}$ -approximation then for each $A \subseteq W$,

$$S(A, W) = 1, \quad (3.2)$$

and

$$\underline{G}(A) = \overline{G}(A). \quad (3.3)$$

Proof. By S -complement condition, we have $S(A, W) = 1 - S(A, \emptyset) = 1$.

For the last part we have

$$\begin{aligned}\overline{G}(A) &= \{x \in U \mid S(T(x), A^c) = 0\} \\ &= \{x \in U \mid S(T(x), A) = 1\} \\ &= \underline{G}(A).\end{aligned}\tag{3.4}$$

□

Assume $G = (U, W, T, S)$ is an $S_{\mathcal{MC}}$ -approximation, then if we define τ as $\{\overline{G}(A) \mid A \subseteq W\}$, (U, τ) becomes a topology. This claim is stated more precisely in the following theorem.

Theorem 3.2. *Let $G = (U, W, T, S)$ be an $S_{\mathcal{MC}}$ -approximation, and τ be defined as*

$$\tau = \{\overline{G}(A) \mid A \subseteq W\}.\tag{3.5}$$

Then (U, τ) is a topology.

Proof. According to definition of topology, (U, τ) should satisfy three conditions.

1. We claim that $\overline{G}(W) = U$ and $\overline{G}(\emptyset) = \emptyset$, so U and \emptyset belong to τ . By definition of $\overline{G}(X)$, we have

$$\begin{aligned}\overline{G}(W) &= \{x \in U \mid S(T(x), W^c) = 0\} \\ &= \{x \in U \mid S(T(x), \emptyset) = 0\} \\ &= U.\end{aligned}\tag{3.6}$$

and

$$\begin{aligned}\overline{G}(\emptyset) &= \{x \in U \mid S(T(x), \emptyset^c) = 0\} \\ &= \{x \in U \mid S(T(x), W) = 0\} \\ &= \emptyset.\end{aligned}\tag{3.7}$$

2. It can be easily seen from the definition of τ that for each $Y_i \in \tau$, where i is in some index set \mathcal{I} , there exists $A_i \subseteq W$ such that $Y_i = \overline{G}(A_i)$. So, it is the case that

$$\begin{aligned}\cup_{i \in \mathcal{I}} Y_i &= \cup_{i \in \mathcal{I}} \overline{G}(A_i) \\ &= \cup_{i \in \mathcal{I}} \{x \in U \mid S(T(x), A_i^c) = 0\} \\ &= \{x \in U \mid \vee_{i \in \mathcal{I}} (S(T(x), A_i^c) = 0)\} \\ &= \left\{x \in U \mid \min_{i \in \mathcal{I}} \{S(T(x), A_i^c)\} = 0\right\} \\ &= \{x \in U \mid S(T(x), \cap_{i \in \mathcal{I}} A_i^c) = 0\} \\ &= \overline{G}(\cup_{i \in \mathcal{I}} A_i).\end{aligned}\tag{3.8}$$

The latter equality is obtained by theorem (2.6), property (1). Therefore $\cup_{i \in \mathcal{I}} \overline{G}(A_i) \in \tau$. Note that U and W are assumed to be finite.

3. We have

$$\begin{aligned}
x \in \cap_{i=1}^n \overline{G}(A_i) &\Leftrightarrow \forall i \in \{1, \dots, n\} \quad S(T(x), A_i^c) = 0 \\
&\Leftrightarrow \forall i \in \{1, \dots, n\} \quad S(T(x), A_i) = 1 \\
&\Leftrightarrow S(T(x), \cap_{i=1}^n A_i) = 1 \\
&\Leftrightarrow S(T(x), (\cap_{i=1}^n A_i)^c) = 0 \\
&\Leftrightarrow x \in \overline{G}(\cap_{i=1}^n A_i).
\end{aligned} \tag{3.9}$$

Therefore $\cap_{i=1}^n \overline{G}(A_i) \in \tau$.

□

If S belongs to $S_{\mathcal{MC}}$, then $\overline{G}(A) = \underline{G}(A)$ for every $A \subseteq W$. From theorems (3.1) and (3.2), the following corollary is obtained, i.e. (U, τ) is also a topology when $\tau = \{\underline{G}(A) | A \subseteq W\}$.

Corollary 3.3. *Let $G = (U, W, T, S)$ be an $S_{\mathcal{MC}}$ -approximation. Define $\tau = \{\underline{G}(A) | A \subseteq W\}$, then (U, τ) is the same topology as in theorem (3.2).*

The set τ has the property that it is closed under complement.

Theorem 3.4. *Let (U, τ) be a topology obtained by theorem (3.2), then (U, τ) is a clopen topology, i.e. every open set is closed.*

Proof. Let $A \subset W$, then by (2.6), $\underline{G}(A) = (\overline{G}(A^c))^c$, and since by theorem (3.1), $\overline{G}(A) = \underline{G}(A)$, so $\overline{G}(A)$ is closed. $\overline{G}(W) = U$ and $\emptyset = U^c \in \tau$, therefore $\overline{G}(W)$ is also closed.

Let Y be a non-empty closed subset of U , so $Y = (\overline{G}(A))^c$ for some $A \subset W$, and since $(\overline{G}(A))^c = \underline{G}(A^c) = \overline{G}(A^c)$, Y is open. It is obvious that \emptyset is open. This concludes the proof. □

4 Enumerating S Functions in $S_{\mathcal{MC}}$

In this section, we suppose that U , W , and $T : U \rightarrow \mathcal{P}^*(W)$ are fixed and then we enumerate all the functions S where $G = (U, W, T, S)$ is an $S_{\mathcal{MC}}$.

Let W be a non-empty finite set, and $f : P(W) \rightarrow \{0, 1\}$ a minimizing function with a single atom, denoted by $\alpha(f)$. If $|\alpha(f)| = 1$, then its only element is denoted by $\mathbf{a}(f)$. The following theorems state the effect of the S -min condition, and a much stricter version of it, on S in terms of its atoms and tend to be very useful tools in counting non-homeomorphic topologies, as we will discuss later.

Theorem 4.1. *Let W be a non-empty finite set, and $f : P(W) \rightarrow \{0, 1\}$ a minimizing function. Then the following are equivalent:*

1. For every non-empty subset A of W ,

$$f(A^c) \leq 1 - f(A). \tag{4.1}$$

2. Either f has a single atom or $f \equiv 0$, i.e. f has no atoms.

Proof. (1 \rightarrow 2) The proof is by contradiction. Suppose that f does not satisfy (2), then $|W| \geq 2$ and each unary subset of W is an atom of f . This way, $f(A) = 1$ for every non-empty subset of W and $f(\emptyset) = 0$. Suppose A is a non-empty proper subset of W . Since A has at least one element and every unary set of W is an atom, then $f(A) = 1$. On the other hand, A^c is non-empty and for the same reason, $f(A^c) = 1$ which contradicts (1).

(2 \rightarrow 1) Suppose $f(A) = 1$ for some non-empty $A \subseteq W$. So $\alpha(f) \subseteq A$, which implies $\alpha(f) \not\subseteq A^c$, hence $f(A^c) = 0$. This obviously yields to (1). \square

Theorem 4.2. *Let W be a non-empty finite set, and $f : P(W) \rightarrow \{0, 1\}$ a minimizing function. Then the following are equivalent:*

1. For every subset A of W ,

$$f(A^c) = 1 - f(A). \quad (4.2)$$

2. f has a single atom and $|\alpha(f)| = 1$.

Proof. (1 \rightarrow 2) By lemma (3.1), $f \not\equiv 0$, so by theorem (4.1), f has a single atom $\alpha(f)$. Now we should show that $|\alpha(f)| = 1$. The proof is by contradiction. Suppose that $|\alpha(f)| > 1$ and $\alpha_1 \in \alpha(f)$. Then $f(\{\alpha_1\}) = 0$. On the other hand, $f(\{\alpha_1\}^c) = 0$ since $\alpha_1 \in \alpha(f)$. This is a contradiction with (1).

(2 \rightarrow 1) Recall that if $|\alpha(f)| = 1$, then $\mathbf{a}(f)$ denotes its only element. Suppose $f(A) = 1$ for some $A \subseteq W$. So $\mathbf{a}(f) \in A$, which implies $\mathbf{a}(f) \notin A^c$, hence $f(A^c) = 0$. This statement can be reversed, so (1) holds. \square

Theorem (4.2) makes it easy to count the number of different functions S in $S_{\mathcal{MC}}$ that can be used to define topologies as stated in theorem (3.2), this number is clearly an upper-bound for the number of different topologies that can be formed as in that theorem.

Theorem 4.3. *Let $G = (U, W, T, S)$ be an S -approximation, and fix the sets U , W , and the relation T . Then the number of different S functions that can be used in order for G to be in the $S_{\mathcal{MC}}$ equals $|W|^{2^{|W|}-1}$.*

Proof. This number can be obtained using the multiplication principle since there are exactly $|W|$ minimizing functions $f : P(W) \rightarrow \{0, 1\}$ that have a single atom $\alpha(f)$ such that $|\alpha(f)| = 1$. \square

Remark 4.4. It is notable that the number obtained above is an upper bound on the number of distinct topologies with fixed U , W , and T .

5 Non-Homeomorphic Topologies of $S_{\mathcal{MC}}$ -approximations

In this section, we would first establish a necessary and sufficient condition so that two topologies (U, τ) and (U', τ') generated by two $S_{\mathcal{MC}}$ -approximations $G = (U, W, T, S)$ and $G' = (U', W', T', S')$ respectively as in theorem (3.2) such that $|U| = |U'|$ and $|W| = |W'|$, are homeomorphic. Then we will use this condition to count such non-homeomorphic topologies.

Lemma 5.1. *Let (U, τ) and (U', τ') be two homeomorphic topologies, $\Phi : U \rightarrow U'$, a homeomorphism between them, $u \in U$, $u' = \Phi(u)$, and $A \in \tau$ an open set containing u , then $\Phi(A)$ contains u' and has the same cardinality as A .*

Proof. It is straightforward. □

For the sake of easier stating the proof of theorem (5.5), we introduce the notion of degree for each element of W .

Definition 5.2. Let (U, τ) be a topology as in theorem (3.2), then the degree of $w \in W$ is defined as

$$\deg_G(w) = |\{u \in U | \alpha(f_{T(u)}) = \{w\}\}|. \quad (5.1)$$

Also, the set W_i , where i is a non-negative integer, is defined as

$$W_i = \{w \in W | \deg_G(w) = i\}. \quad (5.2)$$

Lemma 5.3. *The set of W_i 's, as defined in definition (5.2), is a partition of W .*

Lemma 5.4. *Let $G = (U, W, T, S)$ be an $S_{\mathcal{MC}}$ -approximation that forms a topology as in theorem (3.2). Then for each $w \in W$, $\deg_G(w) = |\overline{G}(\{w\})|$.*

Proof. It is sufficient to show that $\overline{G}(\{w\}) = \{x \in U | \alpha(f_{T(x)}) = w\}$.

$$\begin{aligned} \overline{G}(\{w\}) &= \{x \in U | S(T(x), \{w\}) = 1\} \\ &= \{x \in U | \alpha(f_{T(x)}) \subseteq \{w\}\} \\ &= \{x \in U | \alpha(f_{T(x)}) = w\}. \end{aligned} \quad (5.3)$$

□

Theorem 5.5. *Let $G = (U, W, T, S)$ and $G' = (U', W', T', S')$ be two $S_{\mathcal{MC}}$ -approximations such that $|U| = |U'|$ and $|W| = |W'|$, then their corresponding topologies that are formed as in theorem (3.2) are homeomorphic if and only if for each non-negative integer i , $|W_i| = |W'_i|$.*

Proof. Suppose $|W_i| = |W'_i|$, for all i . We define a function of the form $\gamma : W \rightarrow W'$ such that for each $w_i \in W$ and $w'_j \in W'$, $\gamma(w_i) = w'_j$ implies $\deg_G(w_i) = \deg_{G'}(w'_j)$. Since $|W_i| = |W'_i|$ for all i , we can define a bijective function of this kind. So, from now on, we assume that γ is one-to-one and onto, i.e. a bijection.

Now we define a function $\Phi : U \rightarrow U'$ such that for each $u_i \in U$ and $u'_j \in U'$, $\Phi(u_i) = u_j$ implies that $\gamma(\mathbf{a}(f_{T(u_i)})) = \mathbf{a}(f'_{T'(u'_j)})$, where $f'_{A'}(B') = S'(A', B')$ for every $A', B' \subseteq W'$. By definition of γ it is obvious that Φ can be defined to be a bijection, since

$$\begin{aligned} \deg_{G'}(\mathbf{a}(f'_{T'(u'_j)})) &= \deg_{G'}(\gamma(\mathbf{a}(f_{T(u_i)}))) \\ &= \deg_G(\mathbf{a}(f_{T(u_i)})). \end{aligned} \quad (5.4)$$

Assuming it so, we show that Φ is a homeomorphism between (U, τ) and (U', τ') , where (U', τ') is the topology formed by G' as in theorem (3.2).

It is sufficient to show that Φ^{-1} is continuous, continuity of Φ can be proved in a similar manner.

Let $A \subseteq W$ and $H = \overline{G}(A)$. We need to show that there exists $A' \subseteq W'$ such that $\Phi(H) = \overline{G'}(A')$.

$$\begin{aligned} H = \overline{G}(A) &= \{u \in U \mid S(T(u), A) = 1\} \\ &= \{u \in U \mid \alpha(f_{T(u)}) \subseteq A\} \\ &= \{u \in U \mid \mathbf{a}(f_{T(u)}) \in A\}. \end{aligned} \quad (5.5)$$

We define A' as follows,

$$\begin{aligned} A' &= \{\mathbf{a}(f'_{T'(u')}) \mid u' \in \Phi(H)\} \\ &= \{\gamma(\mathbf{a}(f_{T(u)})) \mid u \in H\}. \end{aligned} \quad (5.6)$$

It is clear by definition of γ and Φ that $A' = \gamma(A)$.

We have

$$\begin{aligned} \overline{G'}(A') &= \{u' \in U' \mid S'(T'(u'), A') = 1\} \quad (* \text{ by definition of } \overline{G'}(\cdot) *) \\ &= \{u' \in U' \mid \alpha(f'_{T'(u')}) \subseteq A'\} \quad (* \text{ all } f' \text{'s are single-atomic} *) \\ &= \{u' \in U' \mid \mathbf{a}(f'_{T'(u')}) \in A'\} \quad (* \text{ all atoms are unary} *) \\ &= \{u' \in U' \mid \exists u \in H \ \mathbf{a}(f'_{T'(u')}) = \gamma(\mathbf{a}(f_{T(u)}))\} \quad (* \text{ by definition of } A' *) \\ &= \{u' \in U' \mid \exists u \in H \ u' = \Phi(u)\} \quad (* \text{ by definition and bijectiveness of } \Phi *) \\ &= \Phi(H). \end{aligned} \quad (5.7)$$

So for every open H in (U, τ) , $\Phi(H)$ is also open in (U', τ') which means that Φ^{-1} is continuous.

Conversely, let Φ be some arbitrary homeomorphism between (U, τ) and (U', τ') . We prove that $|W_i| = |W'_i|$, for all positive integers i . It is clear that in this case, for $i = 0$, $|W_0|$ would be equal to $|W'_0|$ if the equality holds for all other i 's, since $|W| = |W'|$, because W_i 's partitions W and W'_i 's partition W' .

Let $w \in W \cup W'$ be an element with minimal positive degree. We can assume that $w \in W$ without any loss of generality, since Φ^{-1} is a homeomorphism as well. Let u be such an element

of U that $\alpha(f_{T(u)}) = \{w\}$. By property (4) proposition (2.6), $\overline{G}(\alpha(f_{T(u)}))$ is the smallest open set containing u . Let $u' = \Phi(u)$, so $|\overline{G}(\alpha(f_{T(u)}))| = |\overline{G'}(\alpha(f'_{T'(u')}))|$, because the former is the smallest open set containing u and the latter is the smallest open set containing u' and their size must be equal according to lemma (5.1). It must be the case that all elements of $\Phi(\overline{G}(\alpha(f_{T(u)})))$ have the same single element atom, because $\overline{G}(\alpha(f_{T(u)}))$ is a minimal non-empty open set, and so $\Phi(\overline{G}(\alpha(f_{T(u)})))$ is also a minimal non-empty open set. Let's name the element of this atom as w' , we claim that $\deg_{G'}(w') = \deg_G(w)$, and this happens since all elements of $\Phi(\overline{G}(\alpha(f_{T(u)})))$ share one single atom. By lemma (5.4),

$$\begin{aligned} \deg_G(w) &= |\overline{G}(\{w\})| = |\overline{G}(\alpha(f_{T(u)}))| \\ &= |\overline{G'}(\alpha(f'_{T'(u')}))| = |\overline{G'}(\{w'\})| \\ &= \deg_{G'}(w'). \end{aligned} \tag{5.8}$$

So, Φ maps all elements of $\overline{G}(\{w\})$ to all elements of $\overline{G'}(\{w'\})$.

Let's define $G_1 = (U_1 = U - \overline{G}(\{w\}), W_1 = W - \{w\}, T_1, S_1)$ where T_1 and S_1 are induced by U_1 and W_1 from T and S in G , respectively. Let's define $G'_1 = (U'_1 = U' - \overline{G'}(\{w'\}), W'_1 = W' - \{w'\}, T'_1, S'_1)$ similarly and Φ_1 as the induced version of Φ by U_1 and U'_1 . It is easy to verify that Φ_1 is a homeomorphism between G_1 and G'_1 . Continuing with the same procedure, according to finite descent principle leads us to the desired result, since W and U are finite sets. \square

Lemma 5.6. *Let G be defined as in theorem (3.1), then*

$$\sum_{w \in W} \deg_G(w) = |U|. \tag{5.9}$$

Proof. It is straightforward. \square

Let M be a universe. Then we define $\mathcal{T}_{m,n}$ as the set of all topologies (U, τ) made by some $G = (U, W, T, S)$ as in theorem (3.1), such that $U, W \subseteq M$, $|U| = m$, and $|W| = n$. In the following theorem, we will count the number of equivalence classes of $\mathcal{T}_{m,n}$ by $p(m, n)$, where $p(m, n)$ denotes the number of unordered partitions of m into a maximum of n positive integer summands, or equivalently the number of unordered partitions of m into exactly n non-negative integers [17].

Theorem 5.7. *The number of equivalence classes of $\mathcal{T}_{m,n}$ under homeomorphism is $p(m, n)$.*

Proof. According to theorem (5.5), it is sufficient to show that there exists some bijection between partitions of m into n non-negative summands and possible combinations of $|W_i|$'s, since every combination of W_i 's corresponds to a unique equivalence class under homeomorphism. In compliance with lemma (5.6), we can construct such a bijection by defining $|W_i| =$ number of repetitions of the integer i in the partition. It is easily seen that this method defines a bijection. \square

6 Conclusion

S -approximation is a novel tool for studying approximation of uncertain data which is not necessarily described by inclusion relation. We identified a sub-class of S -approximations, called S_{MC} -approximations that have certain topological characteristics that lead to existence of some topologies which are investigated in this paper along with some of their properties. Finally, we have enumerated these topologies up to homeomorphism.

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